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Specific Poincaré map for a randomly-perturbed nonlinear oscillator

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Abstract

The motion of a classical Hamiltonian oscillator, driven by a weak random force, is examined by means of a deterministic approach. We design the specific Poincaré map to find domains of finite-time stability in phase space. The trajectories belonging to these domains remain stable by Lyapunov criteria for the period of mapping T_0 at least. We derive the lower border for the time of phase correlations between close trajectories. It is found that the lifetime of some stable domains significantly exceeds the correlation time of the external force. The randomly-driven Morse oscillator is used as an example.

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1. Introduction

It is well known that chaotic deterministic systems, as well as those under random forcing, exhibit irregular diffusion-like behaviour. Nevertheless, they are commonly studied using different approaches and methods. Nonhyperbolic deterministic systems are known to have mixed phase space partitioned into stable and unstable regions. Motion is almost predictable within stable domains but permits only probabilistic predictions within unstable ones. On the other hand, systems under random forcing are investigated in terms of the purely statistical description, which provides excellent agreement when random deviations of each trajectory from an unperturbed one are numerous. In the statistical approach, the problem of finding an exact solution of the equations of motion is replaced by the problem of finding the respective probability density function in phase space, whose evolution is governed by the Liouville equation. If correlations between close trajectories decay rapidly, the Liouville equation reduces to the Fokker–Planck equation [1]. The time of phase correlations, defined as the timescale of exponential divergence of close trajectories, is commonly assumed to be equal to the reciprocal maximal Lyapunov exponent. If a system under consideration is stochastic, the maximal Lyapunov exponent can be derived using the Furstenberg–Khasminsky formula [2, 3].

The maximal Lyapunov exponent, determined by the invariant measure of a stochastic system under consideration, does not depend on either a realization of the external force or initial conditions. The distribution of the characteristic exponents at finite times, however, is broad, and an appreciable portion of trajectories remain stable on a timescale of several inverse Lyapunov exponents [4] that presumes intermittency-like dynamics. Formation of stable sets in phase space is more prominent in oscillating systems due to resonant interaction of unperturbed motion with a random perturbation [5].

In this paper we look into the stability of a nonlinear oscillator driven by a weak noise from the point of view of deterministic theory. The concept of dynamical chaos is used to describe the system's behaviour at finite times. Our main goal is to link deterministic and probabilistic descriptions and to estimate the horizon of stability for trajectories under random perturbation. The paper is organized as follows. In section 2 we introduce the one-step stroboscopic map, which we entitle as the specific Poincaré map, to determine stable domains in phase space. In section 3, this map is applied to a model of the Morse oscillator. In the final section, we summarize and discuss results of this work.

2. Theoretical analysis

Consider a one-dimensional nonlinear oscillator with the Hamiltonian

$$H = H_0 + \varepsilon H_1(t) = \frac{p^2}{2} + U(q) + \varepsilon V(q)\xi(t), \tag{1}$$

where q is the position, p is the momentum, $U(q)$ is an unperturbed potential with the equilibrium point at $q = q_0$, ε is a small constant, $V(q)$ is a smooth function, $\xi(t)$ is a noise with normalized first and second moments, $\langle\xi\rangle = 0$ and $\langle\xi^2\rangle = 1/2$. For convenience, we make the canonical transformation from the variables (p–q) to the action and angle variables (I–ϑ) [6]. The action and angle variables are introduced by the following formulae:

$$I = \frac{1}{2\pi}\oint p\,\mathrm{d}q, \qquad \vartheta = \frac{\partial}{\partial I}\int_{q'}^{q} p\,\mathrm{d}q, \tag{2}$$

where q' is a coordinate of one of the turning points, and p is written as

$$p = \sqrt{2[H_0 - U(q)]}. \tag{3}$$

The inverse transformation

$$q = q(I, \vartheta), \qquad p = p(I, \vartheta) \tag{4}$$

reduces the Hamiltonian (1) to a sum of the time-independent term and the perturbation

$$H = H_0(I) + \varepsilon V(I, \vartheta)\xi(t). \tag{5}$$

The equations of motion in terms of the canonical action-angle variables take the form [1]

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{\partial H}{\partial \vartheta} = -\varepsilon\frac{\partial V}{\partial \vartheta}\xi(t), \qquad \frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \frac{\partial H}{\partial I} = \omega + \varepsilon\frac{\partial V}{\partial I}\xi(t), \tag{6}$$

where ω is the frequency of unperturbed oscillations. This form is independent of whether additive ($V = \mathrm{const} * q$) or multiplicative noise is imposed. Hereafter, we will consider some single typical realization of noise $\xi(t)$. As any single realization of a random process is deterministic in the sense that at any given moment of time the respective random function has a certain value, the equations of motion can be treated as deterministic ordinary differential equations.

The notion of stability means a weak sensitivity of a solution to the small changes in initial conditions and, therefore, starting points for the stable trajectories belong to some compact sets in phase space. In the case of a periodically-driven deterministic system, existence of these sets issues immediately from the KAM theory. Unfortunately, the KAM theory is useless in dynamical systems driven by random forces. Although each individual realization of a random perturbation can be treated as a deterministic function rather than a stochastic one, the resonances between unperturbed motion and a perturbation are densely distributed in phase space, and there is no nondestroyed invariant curves. It implies all trajectories are unstable in the limit $t \to \infty$, and that makes a deterministic description of long-term dynamics senseless.

Nevertheless, if one seeks to describe motion only within some finite time interval $[0 : T_0]$, 'deterministic' methods are relevant to explore stable sets satisfying the condition of finite-time invariance: *if any set in phase space at* $t = 0$ *transforms to itself at* $t = T_0$ *without mixing, then it corresponds to an ensemble of trajectories which are stable by Lyapunov within the interval* $[0 : T_0]$. According to this statement, stable sets can be found using the specific Poincaré map

$$I_{i+1} = I(t = T_0, I_i, \vartheta_i), \qquad \vartheta_{i+1} = \vartheta(t = T_0, I_i, \vartheta_i), \tag{7}$$

where $I(t; I_i, \vartheta_i)$ and $\vartheta(t; I_i, \vartheta_i)$ are the solutions of equations (6) with initial conditions $I(0) = I_i, \vartheta(0) = \vartheta_i$. The specific Poincaré map can be constructed with any canonically conjugated variables. The basic rule is that values of these variables, calculated at the ith step of mapping, become the initial conditions for the next step. A realization of $\xi(t)$ is one and the same for all steps of mapping. In point of fact, this map is equivalent to a Poincaré map for a system with the Hamiltonian

$$H = H_0(I) + \varepsilon V(I, \vartheta)\tilde{\xi}(t), \tag{8}$$

where $\tilde{\xi}(t)$ is a periodic function consisting of identical pieces of $\xi(t)$ of the same duration T_0

$$\tilde{\xi}(t + nT_0) = \xi(t), \qquad 0 \leqslant t \leqslant T_0, \tag{9}$$

where n is an integer. In this way we replace the original stochastic dynamical system by a periodically-driven one. It should be emphasized that this replacement is valid because we restrict ourselves by considering dynamics within the interval $[0 : T_0]$ only. The key property of the specific Poincaré map follows directly from its analogy with the usual Poincaré map and can be declared in the following way: *each point of a continuous closed trajectory of the specific Poincaré map corresponds to a starting point of the solution of equations (6) which remains stable by Lyapunov at the time* T_0. The inverse assertion is not, in general, true. It will be shown below that the specific Poincaré map provides a sufficient but not necessary criterion of stability.

Topological properties of trajectories of the specific Poincaré map can be studied in the framework of the theory of nonlinear resonance [1]. The functions $V(I, \vartheta)$ and $\tilde{\xi}(t)$ can be decomposed into the Fourier series

$$V = \sum_{l=-\infty}^{\infty} V_l \exp[\mathrm{i}(l\vartheta + \phi_l)], \qquad \tilde{\xi} = \sum_{m=-\infty}^{\infty} \xi_m \exp[\mathrm{i}(m\Omega t + \psi_m)], \tag{10}$$

where $\Omega = 2\pi/T_0$. If $V(I, \vartheta)$ is an analytical function, the Fourier amplitudes decay as $V_l(I) \sim \exp[-\sigma(I)l]$, where σ is the minimal distance between a singularity of $V(\vartheta)$ in the complex plane and the real axis [7]. The Fourier amplitudes for the function $\tilde{\xi}(t)$ decay as $\xi_m \sim m^{-\beta}$, where the parameter β is determined by smoothness of the function $\tilde{\xi}(t)$ (see, for instance, [8] and references therein).

Taking into account (9), and substituting (10) into (6), we rewrite the equations of motion as follows:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{\mathrm{i}\varepsilon}{2}\sum_{l,m=-\infty}^{\infty} lV_l\xi_m\,\mathrm{e}^{\mathrm{i}\Psi}, \qquad \frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \omega + \frac{\varepsilon}{2}\sum_{l,m=-\infty}^{\infty}\frac{\partial V_l}{\partial I}\xi_m\,\mathrm{e}^{\mathrm{i}\Psi}, \tag{11}$$

where $\Psi = l\vartheta - m\Omega t + \phi_l - \psi_m$. The stationary-phase condition $\mathrm{d}\Psi/\mathrm{d}t = 0$ implies the resonances of map (7) $mT(I = I_{\mathrm{res}}) = lT_0$, where $T(I) = 2\pi/\omega(I)$ is the period of unperturbed oscillations. The relation $l{:}m$ defines the order of the respective resonance. It should be noted that an infinite number of resonances $kl{:}km$ ($k = 1, 2, 3, \ldots\infty$) corresponds simultaneously to each resonant action. However, if I_{res} is far enough from the separatrix value, the product $V_{kl}\xi_{km}$ decreases rapidly with increasing k and only the resonances with small l and m can significantly affect trajectories. Thus, if $T_0 > T(I_{\mathrm{res}})$, only the superior term with $l = 1$ should be taken into account in equations (11). Elimination of the resonances with $l > 1$ allows one to describe the motion in the vicinity of I_{res} in the pendulum approximation [1, 7]. Leaving only the resonant terms we can rewrite equations (11) in the form:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \varepsilon V_1(I_{\mathrm{res}})\xi_m\sin\Psi, \qquad \frac{\mathrm{d}\Psi}{\mathrm{d}t} = \omega - m\Omega + \varepsilon\frac{\partial V_1(I_{\mathrm{res}})}{\partial I}\xi_m\cos\Psi. \tag{12}$$

The system of the coupled equations (12) ensues from the universal Hamiltonian of nonlinear resonance [1]

$$H_u = \tfrac{1}{2}|\omega_I'|(\Delta I)^2 + \varepsilon V_l\xi_m\cos\Psi, \tag{13}$$

where $\Delta I = I - I_{\mathrm{res}}$. Its solution describes the phase oscillations near elliptic fixed points $\Psi = 0$ of an isolated resonance. An angular location of the fixed points depends on the random phase ψ_m and, therefore, varies from one realization of $\xi(t)$ to another. A trajectory of map (7), being captured into a resonance, draws a chain-like pattern in phase space. In the stable regime, neighbouring chains are far enough from each other, and the space amid them is filled by non-resonant stable trajectories. Equation (13) gives the rough estimate of the width of a resonance

$$\Delta\omega = \alpha\Delta I_{\max} \simeq 2\sqrt{\varepsilon\alpha V_1\xi_m}, \tag{14}$$

where $\alpha = \mathrm{d}\omega/\mathrm{d}I$ is a characteristic of the nonlinearity of the oscillator. The distance between resonances of the mth and $(m+1)$th orders is equal to $\delta\omega(T_0) = 2\pi/T_0$ and decreases with increasing T_0. If the criterion of Chirikov [8]

$$\frac{\Delta\omega}{\delta\omega} = K \simeq 1, \tag{15}$$

holds, resonances overlap in phase space and the phase oscillations become unstable. The time in order for all resonances in phase space to intersect each other can be estimated from (15) and thus, satisfies the equation:

$$T_{\mathrm{c}} - \frac{\pi K}{\sqrt{\varepsilon\alpha V_1(I_{\min})\xi_m(T_{\mathrm{c}})}} = 0. \tag{16}$$

More exactly, T_{c} is the time of destabilization of low-energy oscillations with minimal values of the action. Let us assume that for $T_0 > T(I)$, the amplitudes ξ_m are expressed as $\xi_m = \xi_1(\beta)m^{-\beta}$, where $m = T_0/T(I)$. Then the solution of equation (16) is the following:

$$T_{\mathrm{c}} = (\pi K)^{2/(2-\beta)}[\varepsilon\alpha V_1(I_{\min})\xi_1]^{-1/(2-\beta)}T^{-\beta/(2-\beta)}. \tag{17}$$

where $I_{\min}$ is the minimal resonant action. If $\xi(t)$ is the delta-correlated white noise process ($\beta = 0$), one yields

$$T_{\mathrm{c}} = \pi K[\varepsilon|\omega_I'|V_1(I_{\min})\xi_1]^{-1/2}. \tag{18}$$

Chirikov's criterion provides a sufficient condition for emergence of global chaos in typical Hamiltonian systems. The time horizon T_c can be treated as the lower border of the time of phase correlations, because stable regions in phase space can persist at $t > T_c$ as islands, whose boundaries are impenetrable for chaotic trajectories of map (7). The surprising corollary from equation (16) is that the fast decay of correlations of the noise does not imply, in general, smallness of T_c. Indeed, high-frequency harmonics of $\xi(t)$ correspond to weak high-order resonances with small widths. If T_0 is large enough and the noise has some given spectral density, the set of the amplitudes ξ_m is the same for different realizations of the noise. Therefore, the time horizon T_c depends weakly on the realization chosen and may be treated as a characteristic quantity of a system under consideration.

3. Numerical results

In this section, we perform a numerical implementation of map (7) for the model of the randomly-driven Morse oscillator that is used to describe the vibrational motion of diatomic molecules. The respective Hamiltonian is the following:

$$H = \frac{p^2}{2} + D(1 - \exp[-q])^2 + \varepsilon V(q)\xi(t), \tag{19}$$

where $D = 1$, $\varepsilon = 0.01$ and $V(q) = \exp(-q - 1)$. For the sake of simplicity, we assume p, q and t as dimensionless variables. The function $\xi(t)$ is modelled as a sum of 10 000 randomly-phased harmonics with frequencies being distributed uniformly in the interval $[0.02\pi : 2\pi]$. The analytical expressions for the action and the angle are the following:

$$I = \sqrt{2}(\sqrt{D} - \sqrt{D - H}), \qquad \vartheta = \pm\frac{\pi}{2} \mp \arcsin\frac{1 - e^{q}(1 - H/D)}{\sqrt{H/D}}, \tag{20}$$

where the upper and the lower signs correspond to positive and negative values of momentum, respectively. The most accessible action I_s corresponds to the separatrix ($H = D$) and is equal to $\sqrt{2D}$. The period of unperturbed oscillations varies from $\sqrt{2}\pi$ at $I = 0$ to an infinity at the separatrix. The function $V(I, \vartheta)$ reads

$$V(I, \vartheta) = \frac{e^{-1}(1 - H(I)/D)}{1 - [H(I)/D]^{1/2}\cos\vartheta}. \tag{21}$$

Decomposing it into the Fourier series, we obtain

$$V(I, \vartheta) = 2\,e^{-1}\left(1 - \frac{H}{D}\right)\left(V_0 + \sum_m \frac{\exp(-\sigma m)\cos m\vartheta}{(1 - \sqrt{H/D})^{1/2}}\right), \tag{22}$$

where σ is given by the formula

$$\sigma = \ln\left(\frac{\sqrt{H/D}}{1 - (1 - H/D)^{1/2}}\right). \tag{23}$$

As it follows from (23), σ vanishes at the separatrix.

We constructed numerically map (7) with different values of T_0. Each individual orbit was integrated over 2000 mapping periods. The topology of the map was found to be similar for all realizations of $\xi(t)$. Figure 1 represents a typical set of maps computed with an individual realization. Phase space is almost stable with small values of T_0 (see figure 1(*b*)). Chaos occurs only in a narrow layer near the separatrix. In the case of $T_0 = 100$, the chaotic layer expands into internal areas of phase space, but low-amplitude oscillations with small values of the action variable maintain stability. Some of them have the form of islands submerged in the chaotic sea. This case is illustrated in figure 1(*b*). Finally, when the step of mapping T_0

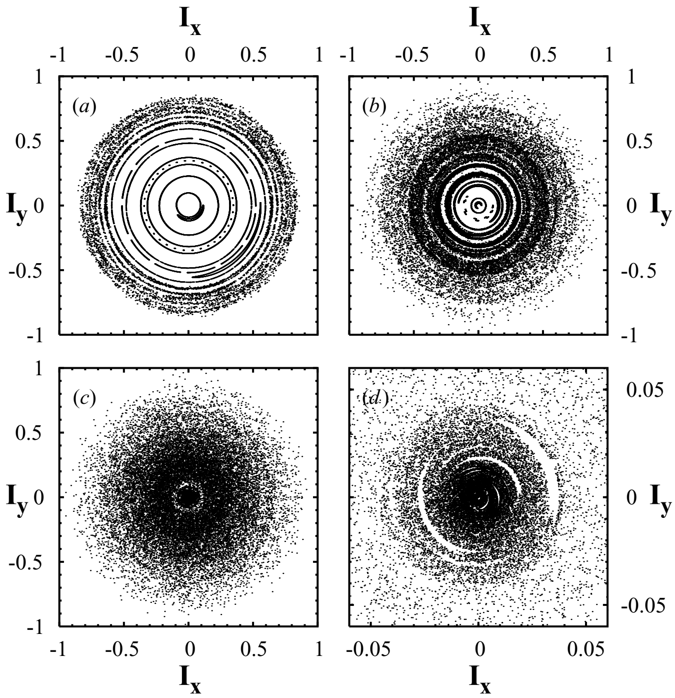

Figure 1. The specific Poincaré map for the randomly-driven Morse oscillator in the polar coordinates action angle with different steps of mapping. (*a*) $T_0 = 10$, (*b*) $T_0 = 100$, (*c*) and (*d*) $T_0 = 1000$. $I_x = (I/I_s)\cos\vartheta$, $I_x = (I/I_s)\sin\vartheta$.

is large enough, phase space is almost chaotic (see figure 1(*c*)), but small islands of stability still survive. Some of them are clearly seen in figure 1(*d*), where the enlarged fragment of figure 1(*c*) is depicted. Several islands are not marked by self-closing curves but can be recognized as unoccupied areas.

Figure 2 represents the analogous set of maps computed with another realization of $\xi(t)$ having the same spectrum. This realization was generated using another random set of phases of harmonics. As was mentioned in the preceding section, locations of the elliptic points of resonances are different in figures 1 and 2. On the whole the macroscopic structure of the specific Poincaré map is one and the same for both realizations.

Modelling of map (7) makes apparent some features of the transport in phase space. Only those trajectories can escape from the potential well, which belong to the chaotic layer near the separatrix. The probability of escaping from the stable area is equal to zero. Moreover, stable internal area is forbidden for chaotic trajectories up to some critical time. Thus the system performs nonergodic properties on finite but relatively long timescales. Manifestations of local nonergodicity are demonstrated in [9], where evolution of patches of passive tracers in a random velocity field is shown.

Formation of stable domains is a consequence of domination of resonances with small l that leads to slow smooth variability of the perturbation term $V(I, \vartheta)\xi(t)$ in phase space. This can be illustrated by the plot that shows by colour modulation values of variations of the normalized action during the oscillator period in the plane of initial values of the action and angle variables [5]. More exactly, we computed a variation of the action between two successive crossings of the line ϑ = const. by a trajectory. The respective plots for the

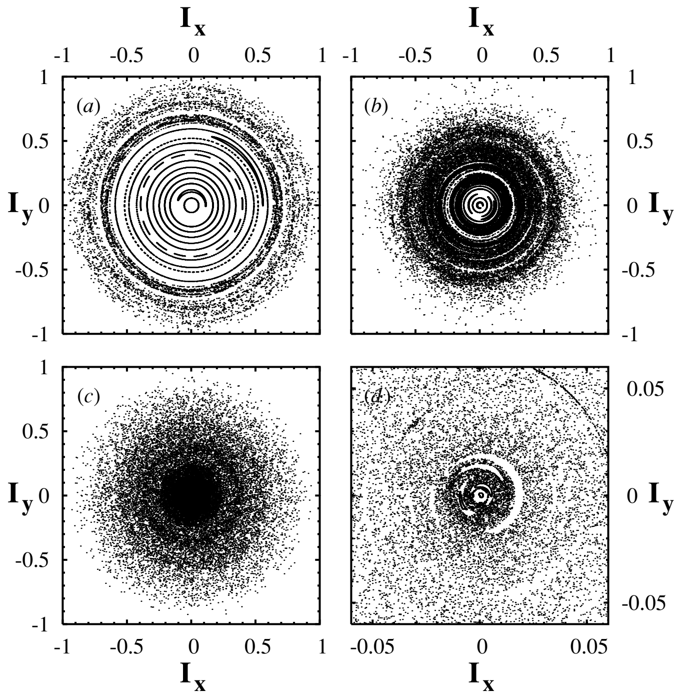

Figure 2. The same as in figure 1, but with another realization of $\xi(t)$.

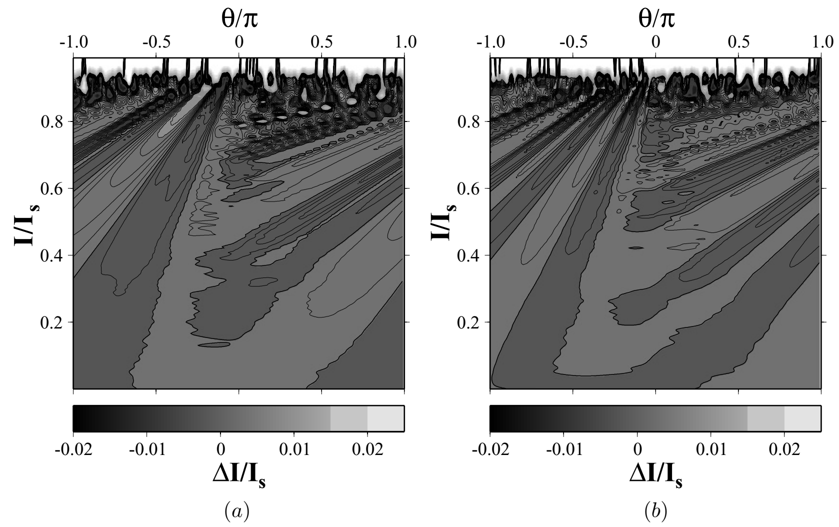

Figure 3. Plot representing variations of normalized action $\Delta I/I_s$ per period of the Morse oscillator in the plane of normalized initial values of the action and angle with (*a*) the same realization as in figure 1, (*b*) the same realization as in figure 2. White colour corresponds to the trajectories crossing the separatrix.

randomly-driven Morse oscillator with different realizations of $\xi(t)$ are presented in figure 3. Both plots demonstrate large alternating 'hills' of positive variations and 'hollows' of negative ones separated from each other by 'zero lines', which correspond to zero variations. If the variation of the action is negative above the 'zero line' and positive below it, then action values of the trajectories with starting points near this line converge to each other. Since 'hills' and 'hollows' are large enough, zones of convergence are expected to be large as well, that anticipates existence of stable domains.

Domination of resonances with small l implies oscillator motion to be essentially responsible to a narrow resonant band in the spectrum of a perturbation, whose frequencies are close to the frequency of an oscillator (this is discussed in detail in [5]). In such a consideration breakdown of stability is connected with increasing an incoherent influence of 'nonresonant' spectral bands, that reveals itself as overlapping of neighbouring resonances of the specific Poincaré map.

4. Conclusion

In the present paper, we offer a robust method of detecting stable domains in phase space of a nonlinear oscillator driven by a random external force. The main advantage of this method is the possibility to analyse purely stochastic systems by means of deterministic theory. In particular, it permits one to detect the fact of the presence of stable domains, which are found to exist for different realizations of a noise with given spectrum.

Existence of long-living stable domains seems to be an important feature in various fields. For instance, a wavefield structure in stochastic waveguides depends on the degree of stability of rays. Interference of the rays, belonging to the stable domains, is constructive, whereas the chaotic rays interfere incoherently [4]. Stable domains can be visualized as compact spots of passive scalars advected by a stationary vortex and an alternating current [9]. Influence of phase space structure on the crossing of a potential barrier is important for the study of stochastic resonance in a double-well potential (see [10] and references therein).

Two remarks about map (7) should be given. First, the total area of the stable domains can be estimated from the specific Poincaré map if only the perturbation is periodic with a period satisfying to the condition $T_p = T_0/n$, where $n = 1, 2, \ldots$. In this case $\tilde{\xi}(t) \equiv \xi(t)$, and the specific Poincaré map coincides with the usual Poincaré map. For nonperiodic perturbations, the total area of the stable domains with a given lifetime T' is larger than it can be exposed by the map with the step of mapping $T_0 = T'$. Second, the map can be constructed for any temporal interval $[t' : t' + T_0]$. Thus, if $\xi(t)$ is a stationary random process, the domains of long-term stability exist at any time moment. Hence it should be proposed that diffusion in phase space, being a sequence of random walks among metastable states, may exhibit anomalous properties and power scaling laws [11]. This topic will be the objective of our forthcoming work.

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